COVER SHEET

Hall, Peter and Wolff, Rodney C (1998) Properties of distributions and correlation integrals for generalised versions of the logistic map. *Stochastic Processes and Their Applications* **77(1):pp. 123-137.**

PROPERTIES OF DISTRIBUTIONS AND CORRELATION INTEGRALS FOR GENERALISED VERSIONS OF THE LOGISTIC MAP

Peter Hall[1] *and* *Rodney C.L. Wolff*[2]

SMS-111-93 CMA-SR30-93

Centre for Mathematics and its Applications
School of Mathematical Sciences
Australian National University
Canberra, ACT 0200, Australia

December 1993

AMS (1991) No.: Primary 60K99, Secondary 60G10

Short title: Logistic map.

Key Words : Chaos, correlation integral, invariant distribution, logistic map.

Home institutions:

1. Centre for Mathematics and its Applications, Australian National University, ACT 0200
2. Department of Statistics, University of Glasgow, G12 8QW, UK

ABSTRACT:

We study a generalized version of the logistic map of the unit interval $(0,1)$, in which the point x is taken to $1-|2x-1|^{\nu}$. Here, $\nu > 0$ is a parameter of the map, which has received attention only when $\nu = 1$ and 2. We obtain the invariant density when $\nu = \frac{1}{2}$, and derive properties of invariant distributions in all other cases. These are obtained by a mixture of analytic and numerical argument. In particular, we develop a technique for combining "parametric" information, available from the functional form of the map, with "non-parametric" information, from a Monte Carlo study. Properties of the correlation integral under the invariant distribution are also derived. It is shown that classical behaviour of this test statistic, which demands that the logarithm of the integral have slope equal to the lag, is valid if and only if $\nu \leq 2$.

Some key words: Chaos, correlation integral, invariant distribution, logistic map.
Short title: Logistic map.
AMS (1991) subject classification: Primary 60K99, Secondary 60G10.

1. Introduction and Summary

Iterative maps of an interval comprise a mathematically important class of examples of chaotic dynamical systems. Their evolving trajectories closely resemble those of a classical time series, yet are strictly deterministic. Unlike more complex, naturally occurring dynamical systems, they are amenable to rigorous numerical analysis and to detailed numerical simulation. Thus, their properties shed light on the way in which more complicated, mathematically intractable maps operate. However, the class of maps that are well understood from a mathematical viewpoint is quite restrictive. In this paper we significantly extend that class by studying a generalized form of the logistic map, which takes x to $f(x) = f_\nu(x) = 1 - |2x - 1|^\nu$ for $x \in \mathcal{I} = (0, 1)$, where $\nu > 0$. The classical case $\nu = 2$, and also $\nu = 1$, have been studied extensively, but other parameter settings have been largely ignored. See May & Oster (1976) for some seminal work in this regard. Isham's (1993) review article covers statistical investigations of such maps.

Invariant distributions of a chaotic map describe the "steady state" of the map, into which the map settles after a long run of iterations starting from "almost anywhere" (in a Lebesgue sense). (In the language of dynamical systems, "almost anywhere" conditions are referred to as "generic".) Thus, they are of considerable statistical interest if the evolution of mapped points is to be compared with that of a stationary time series. See for example Bartlett (1990). For our generalized chaotic map, invariant distributions are known in the cases $\nu = 1$ (where the invariant distribution is uniform on $\mathcal{I}$) and $\nu = 2$ (where it is the Beta $(\frac{1}{2}, \frac{1}{2})$ distribution).

We derive the invariant distribution when $\nu = \frac{1}{2}$; its density is supported on the entire interval $\mathcal{I}$. We also derive properties of invariant distributions when they exist in other cases, along the lines of Hall & Wolff (1993). For example, we show that when $\nu < \frac{1}{2}$, no invariant distribution can be supported on the whole interval $\mathcal{I}$. In the case $\nu > \frac{1}{2}$ we derive properties of the tails of the invariant distribution, including its regular variation at the extremes of $\mathcal{I}$. For general ν's we employ methods based on kernel density estimation to obtain numerical representations of invariant densities where exact mathematical derivation of the density appears impossible. This approach allows us to combine "parametric" information available from the functional form of the map, and "non-parametric" information from a Monte Carlo study of the map, to produce accurate numerical pictures of a range of invariant densities. All these results are described in Section 2.

The correlation integral, D_n, is a commonly used statistic for distinguishing between chaotic and stochastic sequences; see, for instance, Grassberger & Procaccia (1983), Denker & Keller (1986), Wolff (1990) and Hansen (1992). In this context the properties of D_n under the null hypothesis of independence are of critical importance. In Section 3 we investigate those properties in the case where the marginal distribution is the invariant density of our generalized logistic map. It is shown that classical behaviour, in which the logarithm of the correlation integral has slope equal to the lag, is exhibited if and only if $\nu \leq 2$. For $\nu > 2$ the slope equals $2/\nu$ times the lag, which aberration must be taken into account in any statistical test for independence. Departures from the power law for other reasons, as may be due to macroscopic effects, are discussed elsewhere, such as in Smith (1988).

2. Properties of invariant distributions

2.1. Summary

Our aim in this section is to describe, both theoretically and numerically, properties of invariant distributions under the map $x \to f(x)$, where $f(x) = f_\nu(x) = 1 - |2x - 1|^\nu$, $0 < x < 1$ and $\nu > 0$. The distribution of a random variable X is said to be invariant (under f) if both X and $f(X)$ have the same distribution. If the distribution is absolutely continuous with density g then we call g an invariant density.

Broadly speaking, the properties of invariant distributions are as follows. (We do not have proofs of all these assertions in complete generality.) A non-degenerate invariant distribution does not exist for small values of ν, but exists for $\nu > \nu_0$ where $\nu_0 < \frac{1}{2}$ and is close to $\frac{1}{2}$. Only in the case $\nu \geq \frac{1}{2}$ can the invariant distribution have support equal to the interval $[0, 1]$. An invariant density $g = g_\nu$ is bounded for $\frac{1}{2} < \nu < 1$, but is unbounded for $\nu > 1$. In the latter case, $g(x) \sim c_1 x^{(1/\nu)-1}$ and $g(1-x) \sim c_2 x^{(1/\nu)-1}$ as $x \downarrow 0$, where $c_1, c_2 > 0$ are constants depending on ν. In general, $c_1 \neq c_2$, reflecting the fact that g_ν need not be symmetric. The density g_ν is bounded on $(\epsilon, 1-\epsilon)$ for each $\epsilon > 0$. Known special cases include $g_{1/2}(x) \equiv 2(1-x)$, $g_1(x) \equiv 1$ and $g_2(x) \equiv \pi^{-1} x^{-1/2}(1-x)^{-1/2}$, for $0 < x < 1$. For the third case see and Ulam and von Neumann (1947).

In Section 2.2, Propositions 2.1, 2.2 and 2.3 establish theoretical properties of invariant distributions in the cases $\nu < \frac{1}{2}$, $\nu = \frac{1}{2}$ and $\nu > \frac{1}{2}$, respectively. Section 2.3 presents numerical information obtained from a simulation study.

2.2. Theoretical results

Our first result describes the support of an invariant distribution for small values of ν.

PROPOSITION 2.1. *If $0 < \nu < \frac{1}{2}$, and if an invariant distribution exists, then it has no support on the set $(0, x_1) \cup (x_2, x_3) \cup (x_4, 1)$, where $x_1, \ldots, x_4$ satisfy $0 < x_1 < x_2 < \frac{1}{2} < x_3 < x_4 < 1$ and are defined by $f(x_1) = f(x_4) = x_1$, $f(x_2) = f(x_3) = x_3$. (Thus, x_1 and x_3 are fixed points of f.) In particular, the distribution has no support in a non-empty open interval containing $\frac{1}{2}$.*

Note that if $x \in (0, x_1)$ then $f^n(x) \downarrow 0$ as $n \uparrow \infty$; that if $x \in (x_4, 1)$ then $f(x) \in (0, x_1)$; and that if $x \in (x_3, x_4)$ then $f(x) \in (x_4, 1)$. By iterating this argument we may easily identify further intervals, subsets of $(x_1, x_2) \cup (x_3, x_4)$, on which any invariant distribution cannot be supported. Figure 2.3, depicting the Lyapunov exponent of $f = f_\nu$ indicates that no continuous invariant distribution exists for sufficiently small $\nu \in (0, \frac{1}{2})$.

It appears that for such ν, and for almost all $x \in (0, 1)$, we have $f^n(x) \to 0$ as $n \to \infty$.

The next proposition describes the invariant distribution when $\nu = \frac{1}{2}$.

PROPOSITION 2.2. *If $\nu = \frac{1}{2}$ then the only invariant distribution whose distribution function on $(0, 1)$ is analytic, is that with density*

$$g(x) = 2(1 - x), \qquad 0 < x < 1.$$

The corresponding Lyapunov exponent equals $\frac{1}{2}$.

The proof that we give is due to Mark Westcott, who generalised our earlier result that g is the only invariant density which is a polynomial. Let G denote the distribution function of any invariant distribution. By considering the pre-image of a general point $1-x$ we may show that

$$1 - G(1-x) = G\left\{\frac{1}{2}\left(1+x^2\right)\right\} - G\left\{\frac{1}{2}\left(1-x^2\right)\right\}.$$

Thus, with $h(x) = 1 - G(1-x)$ we have

$$h(x) + h\left\{\frac{1}{2}\left(1-x^2\right)\right\} - h\left\{\frac{1}{2}\left(1+x^2\right)\right\} = 0. \tag{1}$$

Expanding the right-hand side of (1) about $\frac{1}{2}$ we deduce that

$$h(x) = \sum_{j=0}^{\infty} h^{(2j+1)}\left(\frac{1}{2}\right)\left\{(2j+1)!\,2^{2j}\right\}^{-1} x^{2(2j+1)}.$$

Replacing x by $x+\frac{1}{2}$ in this formula, expanding, and collecting terms, we deduce that

$$h\left(x+\frac{1}{2}\right) = \sum_{j=0}^{\infty} x^j \sum_{i=[(j+1)/4]}^{\infty} h^{(2i+1)}\left(\frac{1}{2}\right)\binom{4i+2}{j}\left\{(2i+1)!\,2^{6i+2-j}\right\}^{-1},$$

where $[\bullet]$ denotes the operation of taking the integer part. More directly

$$h\left(x+\frac{1}{2}\right) = \sum_{j=0}^{\infty} h^{(j)}\left(\frac{1}{2}\right)(j!)^{-1}x^j.$$

Define $y_j = h^{(2j+1)}(\frac{1}{2})\left\{(2j+1)!\,2^{2j+1}\right\}^{-1}$. Equating powers of x^{2j+1} in the last two displayed formulae we deduce that

$$y_j = \sum_{i=[(j+1)/2]}^{\infty} \binom{4i+2}{2j+1} 2^{-(4i+1)} y_i, \qquad j \geq 0.$$

Equivalently, writing $A = (a_{ij})_{i\geq 0, j\geq 0}$ for the infinite square matrix where

$$a_{ij} = \binom{4i+2}{2j+1} 2^{-(4i+1)},$$

and $y = (y_0, y_1, \ldots)^{\mathrm{T}}$ for an infinite column vector, we have $A^{\mathrm{T}}y = y$.

Each $a_{ij} > 0$; for each i, $\sum_j a_{ij} = 1$; and $a_{00} = 1$. Therefore, A is a stochastic matrix, and 0 is an absorbing state of the corresponding Markov chain. If each other state reaches 0 with probability 1, i.e., if 0 is recurrent, then it follows that the only solution of the equation $A^{\mathrm{T}}y = y$ is $y = (c, 0, 0, \ldots)^{\mathrm{T}}$, for an arbitrary constant c. To show that 0 is recurrent, define $z_0 = 0$ and $z_j = 2j + 1$ for $j \geq 1$, and observe that

$$\sum_{j=0}^{\infty} a_{ij} z_j = (1 - 2^{-4i}) z_i < z_i.$$

This implies that 0 is recurrent (Cox & Miller, 1965, p. 113).

Our final theoretical result in this section describes the tails of invariant densities when $\nu > \frac{1}{2}$.

PROPOSITION 2.3. *If $\nu > \frac{1}{2}$ and g is any invariant density such that $g(\frac{1}{2}) > 0$ and g is continuous at $\frac{1}{2}$, then*

$$g(1 - x) \sim c_2 x^{(1/\nu)-1} \tag{2}$$

as $x \downarrow 0$, where $c_2 = \nu^{-1} g(\frac{1}{2})$. If in addition $g(x)$ is regularly varying at the origin then

$$g(x) \sim c_1 x^{(1/\nu)-1} \tag{3}$$

as $x \downarrow 0$, where $c_1 = \left\{(2\nu)^{1/\nu} - 1\right\}^{-1} \nu^{-1} g(\frac{1}{2})$.

To establish this result, note that by considering the pre-image of a general point $1 - x$ we may prove that

$$g(1 - x) = (2\nu)^{-1} x^{(1/\nu)-1} \left[g\left\{\frac{1}{2}\left(1 + x^{1/\nu}\right)\right\} + g\left\{\frac{1}{2}\left(1 - x^{1/\nu}\right)\right\}\right], \; 0 < x < 1. \tag{4}$$

Letting $x \downarrow 0$ we deduce (2). To obtain (3), replace x by $1 - x$ in (4???), and let $x \downarrow 0$. Noting (2) we obtain, if g is regularly varying with exponent α at the origin,

$$\begin{aligned} g(x) &\sim (2\nu)^{-1} \left[g\left\{1 - (2\nu)^{-1} x + O(x^2)\right\} + g\left\{(2\nu)^{-1} x + O(x^2)\right\}\right] \\ &\sim c_2 (2\nu)^{-1/\nu} x^{(1/\nu)-1} + (2\nu)^{-(\alpha+1)} g(x). \end{aligned} \tag{5}$$

Since g is integrable then $\alpha \geq -1$. If $\alpha > -1$ then by (5), $\alpha = (1/\nu) - 1$ and $g(x) \sim c_1 x^{(1/\nu)-1}$. We show finally that $\alpha = -1$ is impossible.

Suppose $\alpha = -1$, and write $g(x) = x^{-1}L(x)$ where L is slowly varying. Then by (4), as $x \to 0$,

$$
\begin{aligned}
& xg(x) - \frac{1}{2}\left\{1-(1-x)^{(1/\nu)}\right\} g\left[\frac{1}{2}\left\{1-(1-x)^{1/\nu}\right\}\right] \\
= \;& (2\nu)^{-1}x(1-x)^{(1/\nu)-1} g\left[\frac{1}{2}\left\{1+(1-x)^{1/\nu}\right\}\right] \\
& + \left[(2\nu)^{-1}x(1-x)^{(1/\nu)-1} - \frac{1}{2}\left\{1-(1-x)^{(1/\nu)}\right\}\right] g\left[\frac{1}{2}\left\{1-(1-x)^{(1/\nu)}\right\}\right] \\
= \;& O\left[xg\left\{1-(2\nu)^{-1}x + O(x^2)\right\} + x^2 g\left\{(2\nu)^{-1}x + O(x^2)\right\}\right] \\
= \;& O\left\{x^{(1/\nu)} + xL(x)\right\}.
\end{aligned}
$$

Hence, defining $a(x) = \frac{1}{2}\left\{1-(1-x)^{1/\nu}\right\} = (2\nu)^{-1}x + O(x^2)$ as $x \downarrow 0$, we have

$$L(x) - L\left\{a(x)\right\} = O\left\{x^{1/\nu} + xL(x)\right\},$$

whence

$$\left|1 - L\left\{a(x)\right\} L(x)^{-1}\right| = O\left\{x^{1/\nu}L(x)^{-1} + x\right\}.$$

Therefore

$$\log L\left\{a(x)\right\} - \log L(x) = O\left\{x^{1/\nu}L(x)^{-1} + x\right\},$$

and so there exists $\beta > 0$ and $C_1 > 1$ such that

$$|\log L\left\{a(x)\right\} - \log L(x)| \le C_1 x^\beta$$

for all $0 < x < 1/C_1$. Let a^j denote the j'th iterate of a. Then $a^j(x) \downarrow 0$ as $j \uparrow \infty$. For all sufficiently small x, $C_2 x \le a(x) \le C_3 x$ where $0 < C_2 \le (2\nu)^{-1} \le C_3 < 1$. Hence, $a^j(x) \le C_3^j x$ and

$$
\begin{aligned}
|\log L(x) - \log L\left\{a^n(x)\right\}| &= \left|\sum_{j=0}^{n-1}\left[\log L\left\{a^j(x)\right\} - \log L\left\{a^{j+1}(x)\right\}\right]\right| \\
&\le \sum_{j=0}^{\infty} C_1\left\{a^j(x)\right\}^\beta \\
&= C_4 x^\beta,
\end{aligned}
$$

where $C_4 = C_1(1 - C_3^\beta)^{-1}$. Therefore, for some $0 < \epsilon \le 1$,

$$\sup_{n \ge 1,\, 0 < x \le \epsilon} |\log L(x) - \log L\{a^n(x)\}| \le C_4.$$

It follows that $|\log L(x)|$ is bounded as $x \to 0$. Therefore, $xg(x) = L(x)$ is bounded away from zero as $x \downarrow 0$. This contradicts the fact that g is integrable, and so our assumption that $\alpha = -1$ must have been false.

Proposition 2.3 includes the case $\nu = 1$, where it is known that the invariant density g is just the uniform density on $(0,1)$, $g(x) \equiv x$. Here, $f = f_1$ is the so-called "tent map" on the unit interval. Importantly, $\nu = 1$ forms the boundary between cases where the invariant density is bounded in neighbourhoods of $x = 0$ and $x = 1$ (for $\nu \le 1$) and cases where it is unbounded (for $\nu > 1$).

2.3. Numerical results

Recall from formula (4) that

$$g(x) \;=\; (2\nu)^{-1}(1-x)^{(1/\nu)-1}\left(g\left[\frac{1}{2}\left\{1+(1-x)^{1/\nu}\right\}\right] + g\left[\frac{1}{2}\left\{1-(1-x)^{1/\nu}\right\}\right]\right), \qquad 0 < x < 1.$$

By iterating this formula a total of ℓ times we may derive an expression of the form

$$g(x) = \sum_{j=1}^{2^\ell} A_j(x) g\{B_j(x)\}, \tag{6}$$

where A_j, B_j are known functions of x and ν . We shall use equation (6) to combine "parametric" information about g with "non-parametric" information available by Monte Carlo simulation, as follows. Statistical details of this approach, in the special case $\nu = 2$, are discussed in Hall & Wolff (1993).

A sequence of data values $\mathcal{X} = \{X_1, \ldots, X_n\}$ was derived by first "burning in" the map over a run of length 1,000, then iterating the map a further $n = 1,000$ times, then repeating this entire operation $m = 100$ times, thereby obtaining the $N = nm = 100,000$ numbers X_i. To reduce the impact of edge effects, the original sequence $\mathcal{X}$ was reflected in the points 0 and 1, obtaining a new sequence $\mathcal{Y} = \{X_i, -X_i, 2 - X_i;\ 1 \le i \le N\}$, of size $3N$. A kernel estimator of the invariant density g is given by

$$\tilde{g}(x) = \tilde{g}(x; h) = (Nh)^{-1} \sum_{y \in \mathcal{Y}} K\{(x-y)/h\}, \qquad 0 < x < 1,$$

where h denotes bandwidth and $K(x) = (3/4)(1-x^2)I(|x| \le 1)$ is the Epanechnikov kernel. Substituting into equation (6) we derive an alternative estimator,

$$\widehat{g}(x) = \widehat{g}(x; h, \ell) = \sum_{j=1}^{2^\ell} A_j(x)\widehat{g}\{B_j(x)\}, \qquad \ell \ge 1. \tag{7}$$

Figure 2.1 depicts versions of $\widehat{g}$ for various values of $\nu > \frac{1}{2}$. The caption indicates choice of h and ℓ. Figure 2.2 illustrates the invariant distribution function for two values of ν less than and equal to $\frac{1}{2}$, computed as the empirical distribution function of those out of the N values of X_i which were not (absorbed at) 0.

Figure 2.3 illustrates the Lyapunov exponent for the map f_ν,

$$\lambda_\nu = \int_0^1 \{\log |f_\nu'(x)|\}\, g_\nu(x)dx, \qquad \nu > 0,$$

estimated as the value of

$$N^{-1} \sum_{i=1}^{N} \log |f_\nu'(X_i)|. \tag{8}$$

Pointwise error bars, represented as point estimate plus or minus twice the standard deviation of the n values of the average of $\log |f_\nu'(X_i)|$ over m values of i, are also included in the figure. Note that the error bars become very wide as ν decreases below $\frac{1}{2}$. Nevertheless, there is evidence that $\lambda_\nu > 0$ for $\frac{1}{2} - \epsilon < \nu < \frac{1}{2}$ and ϵ sufficiently small. This suggests, although of course does not prove, chaotic behaviour of deterministic sequences derived from the map f_ν when $\frac{1}{2} - \epsilon < \nu < \frac{1}{2}$.

The Lyapunov exponent

$$\lambda_{\nu,\theta} = \int \left\{\log \left|f'_{\nu,\theta}(x)\right|\right\} g_{\nu,\theta}(x)dx \tag{9}$$

for the two–parameter generalized logistic map

$$f_{\nu,\theta}(x) = \theta\left\{1 - |2x - 1|^\nu\right\}, \tag{10}$$

where $\nu > 0$ and $0 < \theta < 1$, may be estimated as in (8), plus a term $\log\theta$. Thus, the behaviour of the Lyapunov exponent in the case of the generalised map is fundamentally the same as that for the one–parameter map. Moreover, it is straightforward to estimate at which values of ν and θ that $\lambda_{\nu,\theta}$ becomes positive. In (9), $g_{\nu,\theta}$ is the invariant density corresponding to $f_{\nu,\theta}$.

3. Properties of the correlation integral

The correlation integral $D(n,p,u)$, defined at (11) below, is frequently used to distinguish between chaotic and stochastic sequences $\{X_i\}$ having the same marginal distribution. When the sequence is stochastic, a graph of $\log D(n,p,u)$ against $-\log u$ should produce a straight line of slope p, for $p \geq 1$; but when the sequence is chaotic, the slopes should be identical for all values of p which exceed the correlation dimension of the map. The map considered here, of the form $X_{i+1} = f(X_i)$ where $f(x) = 1 - |2x - 1|^\nu$, has correlation dimension $d = 1$.

The regression–type argument above is based on calculations which are valid when the marginal density g of the X_i's is bounded, but are not necessarily applicable to other situations. We know from Section 2 that when $\nu > 1$ the marginal density is not bounded, and so the validity of the accepted approach might reasonably be questioned. In the present section we show that the usual approach is valid for $\nu \leq 2$, but that for $\nu > 2$ the slope of the regression line of $\log D(n,p,u)$ on $-\log u$ is $2p/\nu$, rather than p, in the stochastic case. Likewise, smaller slopes are to be expected in the chaotic case, where those slopes will not increase with p for $p \geq d$. When ν is large the slopes for different p's, in the stochastic case, will be relatively close together and hence relatively difficult to distinguish from those in the chaotic case. Thus, our results have important qualitative implications for practical discrimination between chaotic and stochastic behaviour of even simple sequences $\{X_i\}$.

The reason that the dividing line between "classical" and "non-classical" slope properties is drawn at $\nu = 2$, rather than $\nu = 1$, is that it turns out that finiteness of $\int g^2$, rather than boundedness of g itself, is crucial to determining slope in the case of stochastic sequences. The density g is bounded if and only if $\nu \leq 1$, whereas $\int g^2 < \infty$ if and only if $\nu < 2$.

Classical work on properties of the correlation integral has been developed and discussed by Takens (1981), Grassberger & Procaccia (1983), Sauer, Yorke & Casdagli (1991), Wolff (1990), Hansen (1992) and Smith (1992), amongst others.

Motivated by the results described in Section 2 we define $g = g_\nu$ to be a density which is bounded if $\nu \leq 1$, and which satisfies

$$g(x) \sim \left\{(2\nu)^{1/\nu} - 1\right\}^{-1} \nu^{-1} g\left(\frac{1}{2}\right) x^{(1/\nu)-1}, \qquad g(1-x) \sim \nu^{-1} g\left(\frac{1}{2}\right) x^{(1/\nu)-1}$$

as $x \downarrow 0$ if $\nu > 1$. Additionally, assume that for $\nu > 1$ the density g is bounded on $(\epsilon, 1-\epsilon)$ for each $\epsilon > 0$, and continuous and nonzero at $x = \frac{1}{2}$. Since the case $\nu = 2$ is of particular interest, and the invariant density there is known to be $g(x) = \pi^{-1}x^{-1/2}(1-x)^{-1/2}$ (Ulam & von Neumann, 1947), we assume this formula for g in that instance.

Let $X_1, X_2, \ldots$ denote a stationary sequence with marginal density g, let $p \geq 1$ be an integer and define $V_i = (X_i, \ldots, X_{i+p-1})^{\mathrm{T}}$. Write $\|\bullet\|$ for the usual Euclidean metric on p-dimensional space, and put

$$C(n,p,u) = \sum\sum_{1 \leq i < j \leq n-p+1} I\left(\|V_i - V_j\| \leq u\right), \qquad u > 0.$$

The number of terms in this double series equals $\frac{1}{2}(n-p)(n-p+1)$, which prompts the definition of a normalised version of $C(n,p,u)$:

$$D(n,p,n) = \binom{n-p+1}{2} C(n,p,u). \tag{11}$$

We claim that for large n and small u,

$$D \simeq e^c \begin{cases} u^p & \text{if } \nu < 2 \\ (u \log u^{-1})^p & \text{if } \nu = 2 \\ u^{2p/\nu} & \text{if } \nu > 2 \end{cases},$$

where

$$e^c = \begin{cases} v_p \int g^2 & \text{if } \nu < 2 \\ v_p(2\pi^{-2})^p & \text{if } \nu = 2 \\ \left(\left[1 + \left\{(2\nu)^{1/\nu} - 1\right\}^{-2}\right] \Gamma(1-2\nu^{-1})\Gamma(\nu^{-1})\left(\Gamma(1-\nu^{-1})^{-1}\right)^p\right) & \\ \times\, \nu^{-2} g\left(\frac{1}{2}\right)^2 \int\cdots\int_{u_1^2+\ldots+u_p^2 \leq 1} |u_1 \ldots u_p|^{(2/\nu)-1} du_1 \ldots du_p & \text{if } \nu > 2 \end{cases},$$

We consider values of n and u which are such that $\to \infty$, $u \to 0$ and $E(C) \to \infty$. This is the so-called "Normal" context, where C has an asymptotic Normal distribution, as opposed to the "Poisson" case, where $E(C)$ has a finite, non-zero limit and the limiting distribution of C is Poisson. We treat the Normal case here because the regularity conditions assume a simpler form, especially when g is unbounded. Indeed, we need only assume that $E(C) \to \infty$ at a rate which is at least as fast as n^ϵ for some $\epsilon > 0$, arbitrarily small. Equivalently, we suppose that $u_n \downarrow 0$ such that for some $\epsilon > 0$,

$$\left.\begin{array}{ll} n^{2-\epsilon}u_n^p & \text{if } \nu < 2 \\ n^{2-\epsilon}(u_n \log u_n^{-1})^p & \text{if } \nu = 2 \\ n^{2-\epsilon}u_n^{2p/\nu} & \text{if } \nu > 2 \end{array}\right\} \to \infty. \tag{12}$$

The theorem below describes the behaviour of the correlation integral in the case of a stochastic sequence $\{X_i\}$.

THEOREM 3.1. *Assume that $X_1, X_2, \ldots$ are independent and identically distributed with the properties described above, and that the sequence of real numbers $\{u_n\}$ satisfies (12). Then for each real sequence $\{u_n'\}$ satisfying $u_n' > u_n$ and $u_n' \downarrow 0$, we have*

$$\sup_{u_n < u < u_n'} \left| \log D(n,p,u) - c - \left\{ \begin{array}{ll} p \log u & \text{if } \nu < 2 \\ p(\log u + \log\log u^{-1}) & \text{if } \nu = 2 \\ (2p/\nu)\log u & \text{if } \nu > 2 \end{array} \right. \right| \to 0$$

with probability one, for each integer $p \geq 1$.

It may also be shown that $C(n,p,u) - EC(n,p,u)$ is asymptotically Normally distributed. Theorem 3.1 represents a significant refinement of related results of the same type, in that it states explicitly that the convergence to zero of the "remainder term" is available uniformly in values of u. Obviously this is crucial for exploring the approximate linearity of $\log D$ in $\log u$, but it is typically ignored in related work on the classical case where g is bounded.

The remainder of this section is devoted to a proof of Theorem 3.1, which for the sake of clarity we divide into six parts, labelled (a) to (f).

(a) Density of $U = X_1 - X_2$. Let

$$g_1(u) = \int_0^1 g(x)g(u+x)dx$$

denote the density of $X_1 - X_2$. We wish to determine the behaviour of g_1 in the neighbourhood of the origin. When $\nu < 2$, $\int g^2 < \infty$ and so $g_1(u) \to \int g^2$

as $u \to 0$. When $\nu = 2$, $g(x) = \pi^{-1}x^{-1/2}(1-x)^{-1/2}$ and so as $u \to 0$,

$$
\begin{aligned}
g_1(u) &= 2\pi^{-2}\int_0^{1/2} x^{-1/2}(u+x)^{-1/2}(1-x)^{-1/2}\{1-(u+x)\}^{-1/2}\,dx \\
&\sim 2\pi^{-2}\int_0^{1/(2u)} y_{-1/2}(1+y)^{-1/2}dy \\
&\sim 2\pi^{-2}\log u^{-1}.
\end{aligned}
$$

When $\nu > 2$ we have, in notation from Proposition 2.3,

$$
\begin{aligned}
g_1(u) &= \left(\int_0^{1/2} + \int_{1/2}^{1}\right) g(x)g(u+x)dx \\
&\sim (c_1^2 + c_2^2)\int_0^{1/2} x^{(1/\nu)-1}(u+x)^{(1/\nu)-1}dx \\
&\sim (c_1^2 + c_2^2)\int_0^{\infty} x^{(1/\nu)-1}(1+x)^{(1/\nu)-1}dx \\
&= C_3 u^{(2/\nu)-1},
\end{aligned}
$$

say.

Let $U_1, \ldots, U_p$ denote independent random variables with the distribution of $X_1 - X_2$.

(b) Properties of $\pi(u) \equiv pr\left(U_1^2 + \ldots + U_p^2 \leq u^2\right)$. If $\nu < 2$ then as $u \to 0$,

$$
\pi(u) = u^p \int_{u_1^2+\ldots+u_p^2\leq 1}\!\!\!\cdots\int g_1(uu_1)\ldots g(uu_p)du_1\ldots du_p \sim u^p v_p g_1(0)^p = c_4 u^p,
$$

where $c_4 = v_p(2\pi^{-2})^p$; and if $\nu > 2$ it yields

$$
\pi(u) \sim c_4 u^{2p/\nu},
$$

where

$$
c_4 = c_3^p \int_{u_1^2+\ldots+u_p^2\leq 1}\!\!\!\cdots\int |u_1\ldots u_p|^{(2/\nu)-1}du_1\ldots du_p,
$$

with integration over the same limits as above.

(c) Upper bound to $pr(\|V_i - V_j\| \leq u)$ *for* $i < j$. Let $\mathcal{X} = \{X_{i+1}, \ldots, X_{i+p-1}, X_j, X_{j+1}, \ldots, X_{j+p-1}\}$, and put $W^2 = (X_{i+1} - X_{j+1})^2 + \ldots + (X_{i+p-1} - X_{j+p-1})^2$.

Then

$$\begin{aligned}
\operatorname{pr}(\|V_i - V_j\| \le u) &= E\left[I(W \le u)\operatorname{pr}\left\{(X_i - X_j)^2 + W^2 \le u^2 \middle| \mathcal{X}\right\}\right] \\
&\le E\{I(W \le u)\operatorname{pr}(|X_i - X_j| \le u| \mathcal{X})\} \\
&\le \left\{\sup_x \operatorname{pr}(|X_1 - x| \le u)\right\} \operatorname{pr}(W \le u).
\end{aligned}$$

Hence, by induction over p,

$$\begin{aligned}
\operatorname{pr}(\|V_i - V_j\| \le u) &\le \left\{\sup_x \operatorname{pr}(|X_1 - x| \le u)\right\}^p \\
&= \left\{\sup_x \int_{x-u}^{x+u} g(y)dy\right\}^p \\
&\le C \begin{cases} u^p & \text{if } \nu \le 1 \\ u^{pN} & \text{if } \nu > 1 \end{cases},
\end{aligned}$$

where $C > 0$ is a constant not depending on u.

(d) Approximation to $E(C)$. Let $\mathcal{C}$ denote the set of all pairs of integers (i, j) such that $1 \le i < j \le n - p + 1$, X_{i+k} is independent of X_{j+k} for $0 \le k \le p - 1$, and the variables $X_{i+k} - X_{j+k}$, $0 \le k \le p - 1$, are independent. Put $\tilde{\mathcal{C}} = \{(i, j);\ 1 \le i < j \le n - p + 1,\ (i, j) \notin \mathcal{C}\}$. Then the numbers of elements in $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are respectively equal to $n^2 + O(n)$ and $O(n)$. Hence, using (b) and (c) above, we have as $n \to \infty$ and $u \to 0$,

$$\begin{aligned}
E(C) &= \left\{1 + O(n^{-1})\right\} \frac{1}{2} n^2 \pi(u) + O\left\{n \sup_{(i,j)\in\tilde{\mathcal{C}}} \operatorname{pr}(\|V_i - V_j\| \le u)\right\} \\
&= \{1 + o(1)\} \frac{1}{2} c_4 n^2 \left\{\begin{array}{ll} u^p & \text{if } \nu < 2 \\ (u \log u)^{-1})^p & \text{if } \nu = 2 \\ u^{2p/\nu} & \text{if } \nu > 2 \end{array}\right\} \\
&\quad + O\left(\begin{cases} nu^p & \text{if } \nu \le 1 \\ nu^{p/\nu} & \text{if } \nu > 1 \end{cases}\right) \\
&\sim \frac{1}{2} c_4 n^2 \begin{cases} u^p & \text{if } \nu < 2 \\ (u \log u)^{-1})^p & \text{if } \nu = 2 \\ u^{2p/\nu} & \text{if } \nu > 2 \end{cases}, \qquad (13)
\end{aligned}$$

provided either $\nu \le 1$, or $1 < \nu < 2$ and $n^\nu u^{(\nu-1)p} \to \infty$, or $\nu = 2$ and

$n^2 u^p (\log u^{-1})^{2p} \to \infty$, or $\nu > 2$ and $n^\nu u^p \to \infty$. These conditions are automatically satisfied if the quantity in (12) diverges to $+\infty$, which is required by the assumptions of the theorem.

(e) *Error of C about its mean.* Here we shall prove that if $u_n \downarrow 0$ at the rate specified by the theorem then for each $p \geq 1$,

$$\sup_{u_n < u < u_n'} |C(n,p,u)_E C(n,p,u)| \; \{EC(n,p,u)\}^{-1} \to 0 \tag{14}$$

with probability one.

Given $a > 0$ let $\mathcal{U}$ denote the set of all values of u of the form jn^{-a} which satisfy $u_n - n^{-a} \leq u \leq u_n' + n^{-a}$. Since $C(n,p,u)$ is decreasing in u then (14) will follow if we show that for $a > 0$ sufficiently large,

$$T \equiv \sup_{u \in \mathcal{U}} |C(n,p,u) - EC(n,p,u)| \, \{EC(n,p,u)\}^{-1} \to 0, \tag{15}$$

$$\sup_{u \in \mathcal{U}} \left|C(n,p,u) - C(n,p,u+n^{-a})\right| \{EC(n,p,u)\}^{-1} \to 0, \tag{16}$$

$$\sup_{u \in \mathcal{U}} \left|EC(n,p,u) - EC(n,p,u+n^{-a})\right| \{EC(n,p,u)\}^{-1} \to 0, \tag{17}$$

where the convergences in (15) and (16) are with probability one. Result (17) is easily checked, and we shall not go into detail there. Since

$$\begin{aligned} &\left|\left|C(n,p,u) - C(n,p,u+n^{-a})\right| - \left|EC(n,p,u) - EC(u,p,u+n^{-a})\right|\right| \\ &\leq \; |C(n,p,u) - EC(n,p,u)| + \left|C(n,p,u+n^{-a}) - EC(u,p,u+n^{-a})\right|, \end{aligned}$$

and for $a > 0$ sufficiently large,

$$\liminf_{n\to\infty} \inf_{u_n < u \leq u_n'} E\{C(n,p,u)\} \big/ E\left\{C(n,p,u+n^{-a})\right\} > 0,$$

then (16) follows from (15) and (17). Therefore we confine attention to proving (15).

Let $i = (i^{(1)}, i^{(2)})^{\mathrm{T}}$, $I_i = I\left(\|V_{i^{(1)}} - V_{i^{(2)}}\| \le u\right)$ and $\Delta_i = I_i - E(I_i)$. Observe that

$$C(n,p,u) - EC(n,p,u) = \sum_{i^{(1)}<i^{(2)}}\sum \Delta_i,$$

where "$i^{(1)} < i^{(2)}$" denotes "$1 \le i^{(1)} < i^{(2)} \le n-p+1$". Given pairs $i_j = (i_j^{(1)}, i_j^{(2)})$ for $j = 1, 2$, let $\langle i_1, i_2\rangle = 0$ if $|i_j^{(1)} - i_j^{(2)}| \ge p+1$ for $j = 1$ and 2, and $\langle i_1, i_2 \rangle = 1$ otherwise. If $\ell \ge 1$ is an integer then

$$E\left(\sum_{i^{(1)}<i^{(2)}}\sum \Delta_i\right)^{2\ell} = \sum_{i_1^{(1)}<i_1^{(2)}}\sum \cdots \sum_{i_{2\ell}^{(1)}<i_{2\ell}^{(2)}}\sum E(\Delta_{i1}\ldots\Delta_{i2\ell}), \tag{18}$$

such that, for each sequence $i_1, \ldots, i_{2\ell}$ in the summation, each pair i_j has a partner $i_{j'}$ with $j' \neq j$ and $\langle i_j, i_{j'}\rangle = 1$. The pairs $i_1, \ldots, i_{2\ell}$ may be divided among non–overlapping classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$, with the number and structure of the classes depending on the values of $i_1, \ldots, i_{2\ell}$, such that if $i_j, i_{j'}$ are in the same class then $\langle i_j, i_{j'}\rangle = 1$, and if in a different class then $\langle i_j, i_{j'}\rangle = 0$. Necessarily, $r \le \ell$. The total contribution to the right–hand side of (18) from sequences $i_1, \ldots, i_{2\ell}$ which can be divided among just r classes in this way, is bounded in absolute value by a constant multiple (depending on p) of

$$\left\{\sum_{i^{(1)}<i^{(2)}}\sum E(I_i)\right\}^r = \{EC(n,p,u)\}^r.$$

By hypothesis, $EC(n,p,u) \to \infty$, and so, since $r \le \ell$,

$$E\left(\sum_{i^{(1)}<i^{(2)}}\sum \Delta_i\right)^{2\ell} = O\left[\{EC(n,p,u)\}^{\ell}\right].$$

Hence, for any $\epsilon > 0$,

$$\begin{aligned}
\mathrm{pr}\,(T > \epsilon) &\le \epsilon^{-2\ell} \sum_{u\in\mathcal{U}} \{EC(n,p,u)\}^{-2\ell} E\{C(n,p,u) - EC(n,p,u)\}^{2\ell} \\
&= O\left[\sum_{u\in\mathcal{U}} \{EC(n,p,u)\}^{-\ell}\right] \\
&= o\left[n^a\left\{\inf_{u\in\mathcal{U}} EC(n,p,u)\right\}^{-\ell}\right] = O(n^{-2}),
\end{aligned}$$

provided that a is chosen to be sufficiently large. It follows from the Borel–Cantelli lemma that $T \to 0$ almost surely, which proves (15).

(f) Completion. In view of (13) and (14),

$$\begin{aligned} C(n,p,u) &= EC(n,p,u) + \{C(n,p,u) - EC(n,p,u)\} \\ &= \{1 + o(1)\}\, \frac{1}{2} c_4 n^2 \begin{cases} u^p & \text{if } \nu < 2 \\ (u \log u^{-1})^p & \text{if } \nu = 2 \\ u^{2p/\nu} & \text{if } \nu > 2 \end{cases}, \end{aligned}$$

where on the present occasion "$o(1)$" denotes a random variable which converges to zero uniformly in $u_n \le u \le u_n'$, with probability one. Therefore,

$$\begin{aligned} \log D(n,p,u) &= \log\left[C(n,p,u) / \left\{\frac{1}{2}(n-p)(n-p+1)\right\}\right] \\ &= \log c_4 + o(1) + \begin{cases} p \log u & \text{if } \nu < 2 \\ p(\log u + \log\log u^{-1}) & \text{if } \nu = 2 \\ 2p/\nu \log u & \text{if } \nu > 2 \end{cases}, \end{aligned}$$

which completes the proof of the theorem.

ACKNOWLEDGMENT

Helpful discussions with Drs M. Westcott and B.G. Quinn are gratefully acknowledged.

REFERENCES

BARTLETT, M.S. (1990) Chance or chaos? (with discussion). *J. Roy. Statist. Soc. Ser. A* **153**, 321–347.

COX, D.R. & MILLER, H.D. (1965) *The Theory of Stochastic Processes.* Methuen: London.

DENKER, M. & KELLER, G. (1986) Rigorous statistical procedures for data from dynamical systems. *J. Statist. Phys.* **44**, 67–93.

GRASSBERGER, P. & PROCACCIA, I. (1983) Characterization of strange attractors. *Phys. Rev. Lett.* **50**, 346–349.

HALL, P. & WOLFF, R.C.L. (1993) Properties of invariant distributions and Lyapunov exponents for chaotic logistic maps. *J. Roy. Statist. Soc. Ser. B*, under revision.

HANSEN, M.B. (1992) The behaviour of the correlation integral in the non–linear time series case. *Biometrika* **79**.

ISHAM, V.S. (1993) Some statistical aspects of chaos: a review. In *Chaos and Networks: Statistical and Probabilistic Aspects*, Proceedings of a Workshop held at Sandbjerg Manor and the Department of Theoretical Statistics, Aarhus University, 25 April 1992 to 07 May 1992 (O.E. Barndorff–Nielsen, D.R. Cox, J.L. Jensen & W.S. Kendall, eds). Chapman & Hall: London.

MAY, R.M. & OSTER, G.F. (1976) Bifurcations and dynamic complexity in simple ecological models. *Amer. Natur.* **110**, 573–599.

SAUER, T., YORKE, J.A. & CASDAGLI, M. (1991) Embedology. *J. Statist. Phys.* **65**, 579–616.

SMITH, L.A. (1988) Intrinsic limits in dimension calculations. *Phys. Lett. A* **133**, 283.

SMITH, R.L. (1992) Estimating dimension in noisy chaotic time series (with discussion). *J. Roy. Statist. Soc. Ser. B* **54**, 329–352.

TAKENS, F. (1981) On the numerical determination of the dimension of an attractor. In *Lecture Notes in Mathematics* **898**, (D.A. Rand & L.S. Young, eds), 366–381. Springer Verlag: New York.

ULAM, S.M. & VON NEUMANN, J. (1947) On the combination of stochastic and deterministic processes. *Bull. Amer. Math. Soc.* **53**, 1120.

WOLFF, R.C.L. (1990) A note on the behaviour of the correlation integral in the presence of a time series. *Biometrika* **77**, 689–697.

Caption for Figure 2.1. Plots of the invariant density asssociated with the map $f(x) = 1 - |1 - 2x|^{\nu}$ for ν equals (a) 0.9, (b) 1.1, (c) 1.25 and (d) 2. The ordinary kernel estimate (- - - - -) and the first ($\cdots \quad \cdots \quad \cdots$) and second (—) iterates thereof are displayed, having been derived using the relationship given by (7). The values of ν in panels (a) and (b) illustrate behaviour either side of $\nu = 1$, the classical 'tent map'. The value of ν in panel (d) corresponds to the logistic map with parameter 4. All figures are based on series of length 1,000 burned in over 1,000 epochs. The Bartlett–Epanechnikov kernel was used with smoothing parameter 0.2.

Caption for Figure 2.2. Plots of the invariant density asssociated with the map $f(x) = 1 - |1 - 2x|^{\nu}$ for ν equals (a) 0.49 and (b) 0.5. The curve for case (a) was estimated using (7), while that for (b) is $g(x) = 2(1 - x)$. Bootstrap simulations provided an interval estimate of the Lyapunov exponent in case (a); it contained zero. The ordinary kernel estimate (- - - - -) and the first ($\cdots \quad \cdots \quad \cdots$) and second (—) iterates thereof, using the relationship given by (7). All figures are based on series of length 1,000 burned in over 1,000 epochs. The Bartlett–Epanechnikov kernel was used with smoothing parameter 0.2.

Caption for Figure 2.3. Plot of the Lyapunov exponent associated with the map $f(x) = 1 - |1 - 2x|^{\nu}$ as a function of $\nu \in (0.2, 2.0)$. Pointwise intervals represent plus and minus twice the bootstrap standard error after 5,000 replications of calculations made on series of length 1,000 which had been burned in over 1,000 epochs. It was found that the interval estimate of the Lyapunov exponent was to the left of zero, covered zero and was to the right of zero according as $\nu \leq 0.46$, $0.46 < \nu < 0.5$ and $\nu \geq 0.5$, approximately.

Figure 2.1

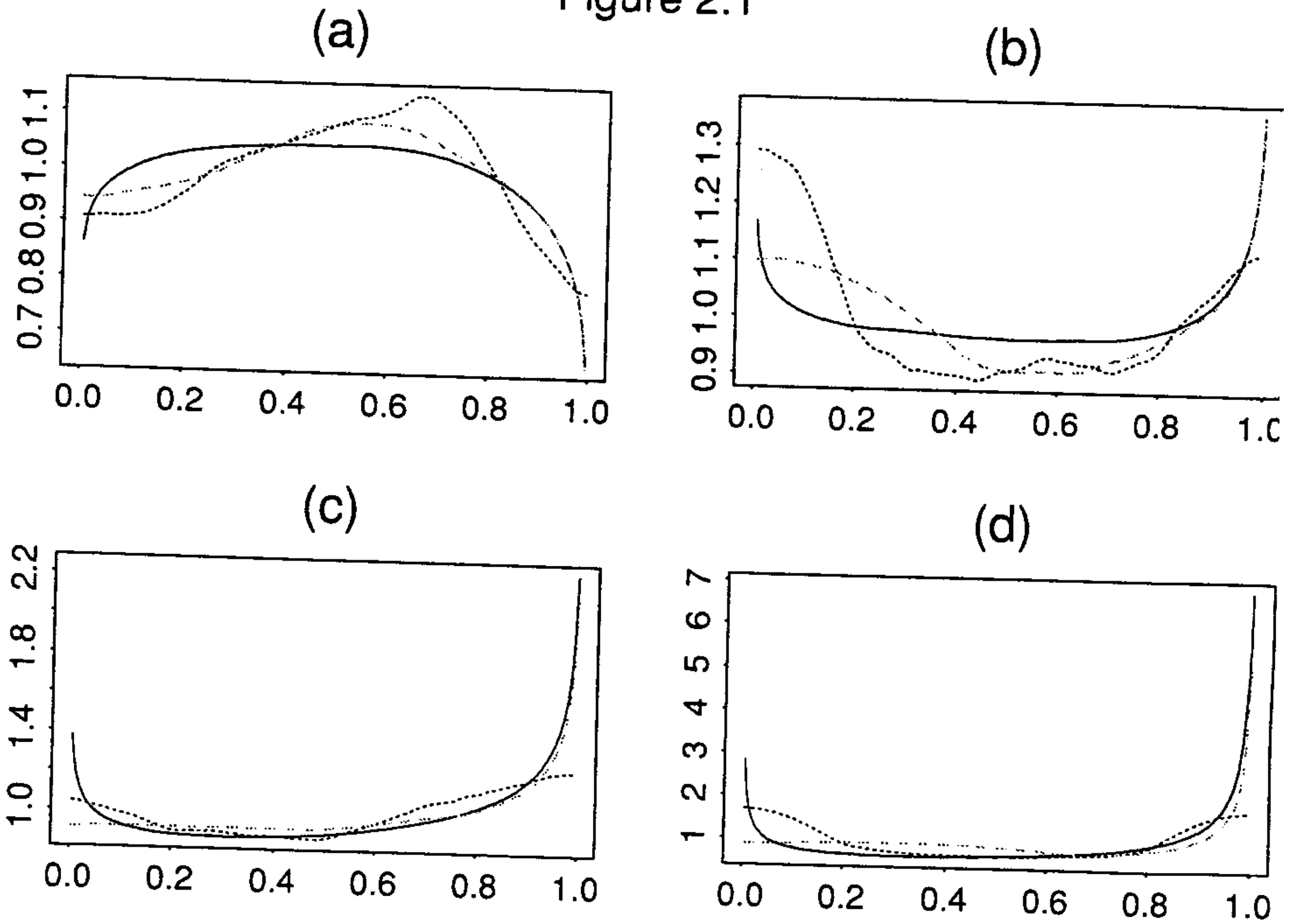
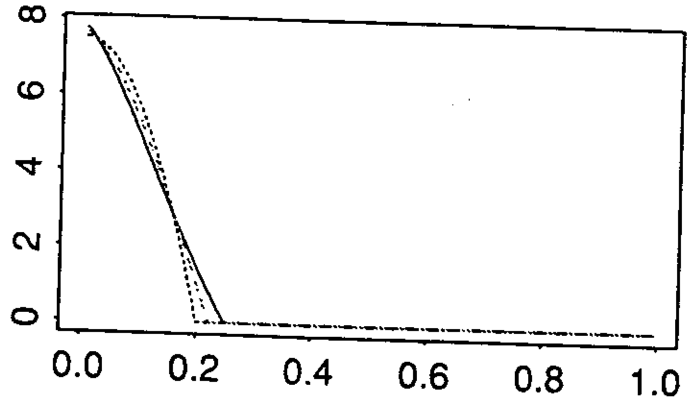

(a)
0.7 0.8 0.9 1.0 1.1
0.0 0.2 0.4 0.6 0.8 1.0
(b)
0.9 1.0 1.1 1.2 1.3
0.0 0.2 0.4 0.6 0.8 1.0
(c)
1.0 1.4 1.8 2.2
0.0 0.2 0.4 0.6 0.8 1.0
(d)
1 2 3 4 5 6 7
0.0 0.2 0.4 0.6 0.8 1.0

Figure 2.2

(a)



(b)

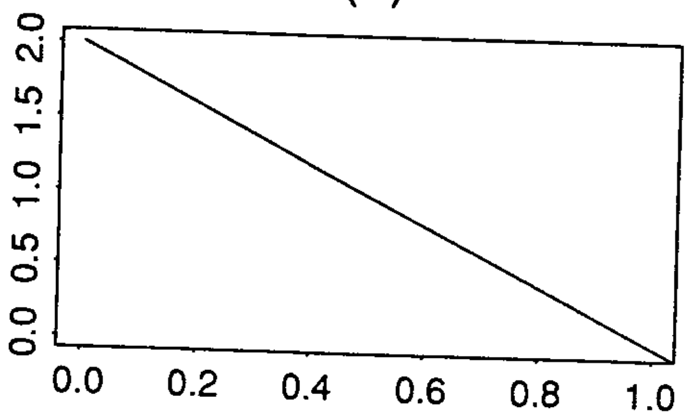


Figure 2.3